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Iconic/propositional representations and graphic presentations: cognitive foundations and a simulation model

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Abstract

Some aspects of the analogical-digital representation in mathematics learning, from a cognitive science perspective, are discussed. In the first part the following issues are addressed: i) differences between *iconic* and *propositional* representation, and problem of their integration; ii) differences and relationships between iconic and mental *representations* and graphic *presentations;* iii) how graphic presentations support representations (such as mathematical situations and problems; mathematical and cognitive operations) that are relevant in mathematical teaching. In the second part, as an example of iconic-propositional integration in the domain of arithmetic problem solving, a simulation model is presented.

Introduction

This paper is divided into two parts. The first section is devoted to make clear, in brief, some conceptual and terminological grounds on which to establish interdisciplinary exchange. In particular, the *analogical - digital* and *representation - presentation* distinctions are examined.

The theoretical premises established in the first section are used also to ground the presentation, in the second section, of part of our current research on analogical and digital representations, including a simulation model of arithmetic problem solving.

Section I - Some conceptual grounds

1. Analogical and digital

In different disciplines, like mathematics and cognitive psychology, concepts are used which look the same but which can be understood very differently. This is the case of the *analogical-digital* distinction, which does not seem to be used consistently in different domains.

In computer science, this distinction reflects the continuous – discrete dimension. From a psychological standpoint, analogical representations reproduce something in an "analog" way, i.e. similar in some aspect to what is being reproduced. Representing by images is analogic in this sense: clearly a dog's image is more similar to a real dog than the letters "d - o - g". Digital representations, on the contrary, are more abstract, they have less to do with things which are represented. They can even be arbitrary at all, like in natural language.

The continuous – discrete dimension, however, is also in the background of the psychological perspective. Analog patterns are closer to the sensorimotor level than digital ones, and the sensorimotor processes are "continuous" by nature.

The difference between analog and digital representations is a classic in cognitive psychology and cognitive science research. It reflects a more general distinction between concrete and abstract representations.

In the educational field there is a huge quantity of research on the different impact of concrete representations as compared with abstract ones. This is, after all, one of the main reasons why mathematics - the kingdom of abstract - asks questions about the use of analogic representations.

If by analogic representation one means a *concrete graphic* representation, then there is also a huge literature about the effectiveness of this kind of representations in various fields, from teaching to science. A well-known article claimed that a diagram is worth more than ten thousand words (Larkin & Simon, 1987).

2. Representations and presentations

From the mathematical standpoint, when one speaks of "representations" it is clear that one is speaking about systems for referring to mathematical entities, which are also abstract entities. It is also clear that one can always put them concretely "black on white" by drawing a graph or writing a formula.

In psychology things are more complicated because psychology, by its very nature, deals with *mental* activities. And also in this case one speaks of representations (mental representations), to mean what in our mind *reproduces* or *stands for* other things. Then the problem arises as what the relation between representations is, in such different senses.

This problem is particularly relevant about the so called *graphic* representations. When we use "representations" of a graphical kind, we are not simply putting on the sheet or on the blackboard representations we have in our head (or, if they are analogic, mental images).

It is not at all granted that there is a direct correspondence between mental representations and external representations we produce on paper, on a blackboard, on a transparency. Even to the introspective evidence, it is clear that they are *constructed*, not copied. They are not produced like a photocopy, but they require effort, sometimes there are mistakes, etc. Nothing grants that they are the faithful reproduction of a pre-existing mental image.

Then, given that the term "representation" may be used so differently in these two senses, to give clarity to our speech, it is appropriate to use two different terms. So we distinguish (following Shanon, 1993) between *mental representations* (iconic, if they are made of images) and *graphicpresentations* (the external ones).

3. Two questions on mental representations

Considered that in psychology we are concerned with mental representations, we shall examine now *mental representations* which are relevant when mathematical knowledge is represented. In particular, we can ask two questions:

- what do they represent? i.e., what are their *contents*?
- how do they represent what they are intended to represent? i.e., what are their *features*? (their codes, their structure…)

4. Mathematical knowledge representation: contents

Mathematical knowledge obviously concerns mathematical entities. Looking more in detail, representations relevant in mathematics belong to the following three types.

1. What in general can be defined *mathematical situations*: aspects of reality that can be mathematicized or which have mathematical relevance: quantities, numbers, idealized objects like geometric objects or sets, or even complex concepts like functions or integrals. The mental tools that we must have to represent mathematical situations should make it possible to refer not only to concepts but also to *relations* between concepts.

2. In second place, there must be the way of representing situations where something is not known, that is *mathematical problems*. A problem is a situation where not only some data are missing, but where the goal is posed to modify the situation by applying some "operators" that enable to acquire missing knowledge. In my opinion, it is very important in teaching to distinguish between the representation of mathematical situations and problems. To represent a situation means to identify aspects that make it relevant from a mathematical standpoint, but this does not mean that it is in itself problematic.

3. A third kind of representations which are relevant from the mathematical point of view are the above-mentioned *operators*. An operator is any procedure whose application allows to transform a situation into a different one. Then operators can be coded and standardized in mathematics (e.g. arithmetic or algebraic operations), but we can have also, more generally, *cognitive* operators (which, for example, make it possible to change representations). Operators may have a specific cognitive representation, of a *procedural* nature.

5. Mathematical knowdlege representation: features

After having examined contents (the "what") of mathematical representations, now it's time to examine their features (the "how"). As we have seen before, one of these features is directly connected to the difference between analogical and digital representations: it is code.

a) code

The question is: how are mathematical mental representations coded? The answer is that mathematical representations, in fact, have no special features in themselves. They are coded like all other "normal" representations.

A well-known general distinction is posed between two kinds of representation codes: verbal - nonverbal. But nonverbal representations are not only sensorial, like visual images or acoustic images, but may be also kinestethic, that means triggered by the "internal sensation of movement" coming from receptors in muscles, tendons, internal organs, etc. The most important in mathematical teaching are *propositional* representations, and visual images or *iconic* representations. (But also kinestethic representations may be relevant, for example in the early stages of acquisition of mathematical concepts: e.g. number as a generalization of the motor pattern of "counting").

Verbal representations are made out of *labels* (symbols or names for mathematical objects or entitites), but they are not an inert collection of labels, symbols or names, because their purpose is to be used to say something about concepts, to express relationships. These relations are expressed by connecting labels in *propositions*, using the natural language or an artificial language like the mathematical or the logical language. This is why we speak of *propositional* representations.

Visual images or *iconic* representations can be used to represent some mathematical entities like sets or geometrical figures, but also to express relations in a concrete form (at most spatial and topological relations, or in a translate form temporal relations).

Iconic mental representations and graphic presentations, however, could not be able to express complex situations and relationships by themselves, without the aid of propositional representations, without being integrated with verbal elements (symbols, captions). A mental image doesn't "speak" by itself, a diagram doesn't express anything if it is not "read" the right way. Most of the effectiveness of analogical re/presentations depends on how they are integrated with digital ones.

The study of such an integration is an interesting research topic, and it's just the focal point of the present paper. But before describing a model of such integration in the domain of arithmetic problem solving, we need a last step in our concept definition. Indeed, we need a basic model of knowledge representation, to ground our analogical-digital integrated model. However, several of such basic models exist, and we are forced to choose one.

b) structure and functions

In fact, many hypotheses and many models about the nature of mental representations have been put forward, and unfortunately no theory or model is universally accepted. The most common are semantic networks and frames (with the main subspecies of scripts and plans).

Such systems are well known and, of course, it's outside the scope of this paper to examine them. But one has to choose in order to find out a knowledge representation system suitable to express the iconic-propositional duality.

Semantic networks (Collins & Quillian, 1969) have nodes (where concepts are coded), connected by archs (which code relations). Their benefit is that there is a unique representation for each concept, but the problem is that relations can only be expressed in a propositional way. For example, in ARCH, a well-known example in Artificial Intelligence (Winston, 1975), it's necessary to identify relevant elements (e.g. columns, architraves, etc.) and relevant relations (like "to support", "to touch", ...) to define the concept of "arch".

Frames (Minsky, 1968), which are also well-known, are much more flexible because they can code together expectations (which are slots to be filled) and new information that fills slots. Among their benefits, they can represent hierarchical knowledge (as an example, a room frame may be included in a house frame, but - in turn - it may include a wall frame, etc.). Frames can also express in a natural way logical relations (predicates and arguments). For example, a predicate like "to give" has a fixed number of arguments ("who" gives, "what" gives, "to whom" gives) which can be considered as frame slots.

Because of these benefits, frames seem a good system for representing mathematical concepts. But are they also good in order to express the iconic-propositional duality which is our main interest here? In the second part of this paper, I will discuss a basic model of mathematical problem solving based on frames, in order to highlight some shortcomings of it and to show how a frame representation system can be adapted to express iconic representations.

Section II - A simulation model using iconic representation

1. The Kintsch & Greeno model

As we have seen, frames seem a good system for representing mathematical concepts. They have been used only in a propositional form, but I will try to show how they can also be used as iconic frames.

A basic model of mathematical problem solving using propositional frames has been proposed by Kintsch and Greeno (1985). The representation structures used in KG model are three kind of frames: propositional, general set, single set. **Propositional** frames represent proposition elements (what happens: e.g. to have, to give…).

Examples of propositional frames are:

HAVE (x, y)
HAVE-ALTOGETHER (x_1, x_2, y)
GIVE (x_1, x_2, y)

Since arithmetic problem understanding always concerns sets, two basic **set** frames are devised: general set and single set. **General set** frames represent relations among sets and are cued by some particular propositional frame (e.g. when the propositional frame is "to give", this matches a transfer set, where one has to represent a starting set, a transferred set, and a resulting set).

Examples of general set frames are:

TRANSFER frame (S_1, S_2, S_3)
slots: start set (S_1)
transfer set (S_2)
result set (S_3)

PART-WHOLE frame (S_1, S_2, S_3)
slots: subset (S_1)
subset (S_2)
superset (S_3)

Single set frames represent properties of a specific single set (which elements are inside a set, how many, of what kind, etc).

This is an example of a single set frame.

MARBLE set properties slots:

object (the name of set elements.:e.g. "marbles")

quantity (#, or "some", "how many"...)

specification (who owns it, when, ...)

role (is it a subset? transferred set? ...)

In order to make the KG model work, also other representation structures are needed: some strategies ("**action schemata**" represented as production systems) must be used to trigger the appropriate frames and to fill slots correctly with information from the text.

As an example, the TRANSFERSET procedure sounds like this: if there is a GIVE frame, then call TRANSFERSET to assign roles to sets obtained from text, based on knowledge like "a set owned by the individual who gives, is the starting set", etc.

Arithmetic operators are also represented as production systems (e.g. if there are a *superset* and a *subset* whose quantities are known, and another *subset* whose quantity is unknown, then apply SUBTRACT).

It is outside the scope of the present paper going too in detail into the KG model, but this overview should have given an idea of how complex it is.

In spite of its complexity, the KG model has several shortcomings. The first - and most important - is that it only accounts for the propositional aspect of representation, completely neglecting the iconic aspect. In natural representation, however, propositional and non-propositional aspects are mixed, particularly where problem solving in children is involved. Everyone knows that children are not able to perform operations in a completely abstract way. A complete account of arithmetic problem solving cannot study separately propositional and iconic representations, but must describe how they are integrated.

Secondly, this model uses the concept of "set" as a primitive: frames and comprehension strategies have sets as arguments, but how such sets arise, and are in turn represented, is not explained.

Thirdly, sets which have elements in intersection are considered as separated.
E.g. the TRANSFER frame requires 3 sets (starting set, transferred set, resulting set), but the model lacks the information (essential to the very idea of "transfer") that these sets include, in part, the same elements. To obtain this information, an additional frame is necessary to represent the "part-whole" relationships.

Moreover, when the problem representation is complete, the arithmetic operation is chosen only by pattern-matching ("which sets are there? well, select this operation"); there is no reasoning, the solution seems not really linked to a problem understanding.

2. Towards a simulation model of arithmetic problem solving using iconic schemata

The basic reason of similar shortcomings is simply that iconic representation is not a pictorial embellishment or something that one adds to make things easier, but is a form of representation that takes place in parallel, or is integrated with, propositional representation.

Now I shall describe a different model of arithmetic problem representation, that tries to take advantage of benefits of frame representation systems, but also to account for integration between iconic and propositional aspects.

It is a simulation model based on **iconic frames**, that is frames whose elements (the fixed part and the variable part as well) are iconic. The basic idea is to have frames with digital tokens, which however are manipulated in analogic fashion. This way, they can express in a non-propositional way certain relationships (of course the ones that can be expressed this way, e.g. topological relations).

Let's take for granted the idea, central in the KG model, that **sets** are the basic representation in the domain of arithmetic problem comprehension (I don't want to discuss here if such a presupposition is well grounded). I will start from this structure to illustrate how **iconic frames** can be implemented.

It's easy to accept that the psychological definition of a set is not the same as the mathematical one. The only requirement of a set from a mathematical standpoint is that one must be able to specify precisely which elements belong to it. From the psychological point of view, however, the grouping criterion necessarily makes use of *conceptual* categories: a set of 3 apples, 2 bricks, and 4 books has a mathematical sense but no psychological sense.

Then, using a label to represent sets cannot be the starting point, but the ending point, because one can only label a set after having recognized some *conceptual* homogeneity.

Using LISP syntax, a set of marbles may be represented in two different ways:

(SETQ SET (QUOTE MARBLES)) *propositional*
(SETQ SET (QUOTE m m m m m m)) *iconic* (where the number of 'm's is arbitrary)

The change is not only formal. This notation is a first step to allow our model to manipulate lists representing sets as if they held icons. This is somewhat similar to elements of mental models (Johnson-Laird, 1983), since these tokens are both symbols and icons. As symbols, they are manipulable and stand for something else. As icons, their mutual relationships are constrained to be structurally analog to the represented state of affairs.

However, there are substantial differences between iconic frames and mental models. Mental models are a valuable advancement towards integration between propositional and iconic aspects, but they can have explicative value (at least in the domain of problem solving) only if it is made clear how they are constructed from images and/or propositions, and why they work in facilitating solution. A graphic presentation (and a corresponding iconic representation) prerequisite is certainly that it be a true model, i.e. exhibit clear analogical relationships with a situation. However, this is not enough because the relevance of such relationships can only be established by procedures that construct or use representations, by identifying relevant elements and relations. Such procedures are different from manipulation procedures that are fundamental in problem solving: they are in a sense more basic because concern representation, not solution, actions. These actions require dynamical models and therefore, differently from mental models, iconic frames have slots, default assignments, and other features typical of frames.

To explain how our model works, it is useful to make a comparison with the implementation of a propositional representation in the original KG model.

Propositional representation of problem sentences in the KG model	
Sentence	Propositional representation
S1: Joe had 3 marbles	TIME1 (HAVE (J, 3m))
S2: Then Tom gave him some more marbles	TIME2 (GIVE (T, J, (SOME m))
S3: Now Joe has 8 marbles	TIME3 (HAVE (J, 8m))
S4: How many marbles did Tom give him?	GOAL (TIME2 (GIVE (T,J,xm)))

For each sentence there is a time specification and a specific frame is instantiated, i.e. its slots are filled with matching information extracted from the text. For example, in the GIVE frame, the 3 slots are filled with T (Tom), J (Joe), and m (marbles), along with quantifiers (a number or expressions like "some", which later will be replaced by question marks or "x" (unknown quantities).

In the iconic-frame model each sentence activates a propositional representation, but also an iconic one, which substantially explains the problem model construction and the solution. The new idea is that abstract and concrete representations live together in the same representation model.

The first sentence

*Joe **had** 3 marbles*

calls the HAVE iconic-frame, whose **general** form is shown in table 1. This represents that a person p has a number N of objects O, at time t. The set of objects is represented by an arbitrary number of tokens, because number is part of the *propositional* representation (considering that there are *number* slots and *time* slots, this representation is not fully iconic nor fully propositional, but integrated).

Table 1 - HAVE iconic frame (**general** form)
HAVE (O,p): TIMEt (N (Op Op Op Op))

In its **instantiated** form (table 2), objects are replaced with M for marbles, time is TIME1, number is 3, and tokens are Mj (refer to Joe's marbles set, marbles belonging to Joe).

Table 2 - HAVE iconic frame (instantiated form)
HAVE (M,j): TIME1 (3 (Mj Mj Mj Mj))

The sentence S2

Then Tom ***gave*** *him some more marbles*

contains the key-information in order to understand the problem. Now the GIVE iconic frame is triggered, whose general form is in table 3.

Table 3 - GIVE-passive iconic frame (**general** form)		
TIMEt	(N1 (Op_1 Op_1 Op_1 Op_1)	(N2 (Op_2 Op_2 Op_2 Op_2))
TIMEt+1 / EQ	[N (N1 (Op_2 Op_2 Op_2 Op_2)	(N2 (Op_2 Op_2 Op_2 Op_2))]
TIMEt+1 / EQ	[N (Op_2 Op_2 Op_2 Op_2	Op_2 Op_2 Op_2 Op_2)]

This fact (to GIVE) must be represented at two times: at a time t, there are 2 sets of objects which belong to 2 different persons. Objects that at time t belonged to p_1, at time t+1 belong to p_2. (Why this frame is called GIVE-passive is explained below, for the moment consider it as simply GIVE).

Remark that the resulting representation can be expressed in either of two alternative ways, which we call **equivalent representations.** In fact it is assumed that, in order to solve the problem, the subject should be able to easily shift between two equivalent representations. To shift between two representations means that they may be alternatively put in the workmemory and compared.

In the example (see table 4), the representation of two sets **containing only objects that belong to the same individual** is equivalent to the representation of a single set (see Op_2s or Mjs in the EQ lines).

Table 4 - GIVE-passive iconic frame (**instantiated** form after S2)		
TIME2	(? (Mt Mt Mt Mt)	*(3 (Mj Mj Mj Mj))*
TIME3 / EQ	**[? (? (Mj Mj Mj Mj)**	(3 (Mj Mj Mj Mj)) **]**
TIME3 / EQ	**[?** (Mj Mj Mj Mj	Mj Mj Mj Mj) **]**
The part in *italic* shows what is known from the sentence 1, the part in **bold** what is the new information coming from sentence 2.		

The expression "*some* more marbles" is represented by a (bold) question mark in the slot referring to new Mjs.

Another thing to be noted is that this representation implicitly codes also some inferences, like the information that at TIME2 Tom also must had some marbles, which is not explicitly included in the text, but is relevant in order to solve the problem.

The sentence S3

Now Joe has 8 marbles

gives a missing information (the number 8) which was already expected and fills a slot in the very same frame (see table 5), differently from the propositional representation, where a different HAVE frame had to be called to represent the fact that now Joe has 8 marbles.

Table 5 - GIVE-passive iconic frame (**re-instantiated** form after S3)		
TIME2	(? (Mt Mt Mt Mt)	*(3 (Mj Mj Mj Mj))*
TIME3 / EQ	**[8** *(? (Mj Mj Mj Mj)*	*(3 (Mj Mj Mj Mj))* **]**
TIME3 / EQ	**[8 (Mj Mj Mj Mj**	**Mj Mj Mj Mj)]**
The part in *italic* shows what is known from the previous sentences, the part in **bold** what is the new information coming from sentence 3.		

The sentence S4

How many marbles did Tom give him?

sets a GOAL to be attained (i.e. what is the missing information we want to know). Even this missing information can also be coded inside the **same** frame (table 5). Reasoning back in time, about quantities referring to previous times, is easy because a single frame is used.

The unknown set is marked in a special way and denoted as GOAL (i.e. the goal is set to find the number to be placed there, see table 6).

Table 6 - GIVE-passive iconic frame (**re-instantiated** form after S4)		
TIME2	**(GOAL (Mt Mt Mt Mt)**	*(3 (Mj Mj Mj Mj))*
TIME3 / EQ	*[8* **(<u>GOAL</u> <u>(Mj Mj Mj Mj)</u>**	*(3 (Mj Mj Mj Mj))]*
TIME3 / EQ	*[8 <u>(Mj Mj Mj Mj</u>*	*Mj Mj Mj Mj)]*
The part in *italic* shows what is known from the previous sentences, the part in **bold** what is the new information coming from sentence 4. <u>Underscored</u> parts are put in correspondence (see text).		

This set appears twice, representing firstly "Tom's marbles" and then "Joe's marbles (received fromTom)". These sets can be put *in correspondence.*

To put **in iconic correspondence** means that a correspondence is established in workmemory between different representations of a same state of affairs. To establish an iconic correspondence means to represent things differently but to be aware, by icon inspection, that they are the same by some other perspective. We guess that icon inspection may be accomplished by repeatedly and alternatively putting alternative representations in the workmemory, in order to make a coarse evaluation of their role in the general situation. In general, two sets can be put in iconic correspondence if one of these conditions is true:

- if one of them is resulting from a transformation operated on the other
- if, by frame structure, they are equivalent sets

In the GIVE frame, the set of underscored <u>Mj</u>s results from the direct transformation of the underscored <u>Mt</u>s. Between these two representations, then, a correspondence in workmemory can be established.

The solution is not attained by a mechanical activation of some procedure matching the situation (as in the KG model) but inferences are drawn, and upon these inferences the arithmetic operation is selected.

The two TIME3 equivalent representations, which are alternative representations of separation and union of parts, are alternatively shifted in workmemory; finally the representation of the set whose quantity is unknown (the one marked as GOAL) is left focused in workmemory. The system tries to put in correspondence elements of this GOAL representation with the equivalent representation. If it does not succeed in completing the correspondence (there are remaining elements), then the focused GOAL set is a subset and the resolutive operation is subtraction of the smaller quantity from the greater. On the contrary case, the GOAL set is the superset and operation is addition. This procedure is different from KG model, where the role of superset or subset is assigned in a computational, not natural, fashion. As we have seen, that model used a production system, with rules of this kind: "If a set has the start-set role in the TRANSFER schema, then consider it as a subset - The transferred set is also a subset - The resulting set is a superset". The solution procedure only inspects in a table which quantities are known and which not, without really coming to a problem understanding.

The general form of GIVE depicted in table 3 is only a partial representation of a possible state of affairs concerning "giving". In fact, one optional condition should be taken into account in a complete model, that is the fact that "given" objects may or may not be the *only* sets belonging to a same individual. In the previous problem, it is explicitly stated that Tom gives "some" marbles to Joe, but we are not told if Tom has other marbles left. This could be relevant if one had asked "How many marbles does Tom have now?". In other cases, the quantity is clearly fixed: S1 states that "Joe had 3 marbles" and supposing that he had more marbles would simply contradict this sentence. In order to have only a single general GIVE frame, a frame where *both* characters (Tom and Joe) had a two-part set, one fixed and one optional, should be devised. However, such a model would be too complex and it does'nt look natural.

A better way of capturing this general situation is to consider that, in fact, the GIVE frame may be called in two different cases: when some individual's possession becomes *enriched* as a consequence of (passive) being given, and when some individual's possession (the focused character; relative information is always at TIME1) becomes *impoverished* as a consequence of (active) giving.

A more refined general form of the GIVE frame, then, must distinguish between two cases: the GIVE-passive (table 3) case when the act of giving has the effect of augmenting the original set of objects (the one about whom prior information is available) and the GIVE-active case (table 7) when part of the original set is used in giving.

Table 7 - GIVE-active iconic frame (**general** form)		
TIMEt	[N (Op$_1$ Op$_1$ Op$_1$ Op$_1$	Op$_1$ Op$_1$ Op$_1$ Op$_1$)]
TIMEt+1 / EQ	[N (N1 (Op$_1$ Op$_1$ Op$_1$ Op$_1$)	(N2 (Op$_1$ Op$_1$ Op$_1$ Op$_1$))]
TIMEt+1 / EQ	(N1 (Op$_2$ Op$_2$ Op$_2$ Op$_2$)	(N2 (Op$_1$ Op$_1$ Op$_1$ Op$_1$))

Here is an example where this frame is used: Joe had 8 marbles; then he gave 5 marbles to Tom; how many marbles does Joe have now?

In this case, at a time t, only one set is considered, of objects which belong to one person. This set may be split into two parts, of objects that at time t belonged to p_1, but at time t+1 belong to p_2, and objects that at time t+1 continue to belong to p_1. The cognitive operation here is different than in the previous (GIVE-passive) case, because in that case a superset was to be constructed from represented subsets; in the GIVE-active case, at TIME2 a superset must be decomposed into two different subsets (which might be labelled as "given" and "not given" parts). One of these appears twice, and - as previously explained - the key to the solution is to put in correspondence these two equivalent sets. An empirical question is whether problems concerning both situations (giving and being given) are equally difficult; using iconic frame-based simulation can help determine critical points in representing such situations.

The described model works also with more complex situations. In table 1 (HAVE frame) Op are objects included into a specified category (defining attribute: *possession* by a person p). The defining feature could be of a different nature (e.g. color, size, etc.), but since the relevant attribute for HAVE is possession, if the color of marbles had been also specified, then a *partition* space for representing more sets ought to be called. If the problem text specifies that "Joe had 3 *red* marbles", then presumably something will be stated later about non-red marbles. Such more complex situations would be representable in a natural way in our model. In table 8, Op (objects belonging to a person p) are splitted into Opx (having the feature x, or, which is the same, belonging to the category x)and Op-x (where the feature # x stands for non-x), which may or may not exist (the subsequent problem text will make clear if the same person p has non-red marbles, or someone else possesses them, etc.).

Table 8 - HAVE iconic frame with partition
HAVE (O,p): TIMEt (N (Opx Opx Opx Opx) (Op-x Op-x Op-x Op-x))

Our model works also with all problems considered in the KG model. For the "combine" problems the basic frame is HAVE-ALTOGETHER (table 8).

This frame is very simple: these are equivalent representations of two separate parts, which can be seen as joined as well, at the same time.

For "compare" problems the basic iconic frame is HAVE-MORE (table 9). This is more complicated since a set has to be broken into two subsets. Parts can be distinguished because, putting the two subsets in correspondence with another set, one part is corresponding but the other is not.

Table 9 - Combine problems

Example:

Joe and Tom have 8 marbles
Joe has 3 marbles
How many marbles has Tom?

HAVE-ALTOGETHER iconic frame

TIMEt /EQ (Op1 … Op1) (Op2 … Op2)

TIMEt /EQ (Op1 … Op1 Op2 … Op2)

In table 9, a unique set of objects Op_1 (belonging to p_1) is divided into two parts: one corresponds to the set of objects Op_2 (belonging to p_2), the other does not. This last set holds the quantity of objects that p_1 has MORE, or - which is the same - that p_2 has LESS. In the case that we call the situation as "having less", elements are marked with an asterisk to emphasize that they are non-existent objects, which must be imagined.

Table 10 - Compare problems

Example

Joe has 8 marbles.
Tom has 5 marbles less than Joe
How many marbles has Tom?

HAVE-MORE iconic frame

TIMEt /EQ	(Op_1 … Op_1	… Op_1 … Op_1)
TIMEt /EQ	((Op_1 … Op_1)	(Op_1 … Op_1)
	(Op_2 … Op_2)	
		MORE (Op_1 … Op_1) LESS (Op_2* … Op_2*)

In both cases of having more and having less the same frame is used, and, like before, the comprehension comes from shifting between equivalent representations of the same situation.

3. Discussion

The model we propose is simpler than KG but more powerful. Complicated strategies to assign roles to sets are no longer neeeded. These roles emerge automatically from iconic representation. Moreover, less frames are needed (for problems with inverse operation, representation is unique: have-more-than and have-less-than, give and receive, etc.). Frames themselves are also simpler.

When scaling-up a model to more complex problems, propositional frame complexity grows with problem complexity. E.g. for GIVE it could be necessary to specify some prerequisites like the fact that p_1 owns what he gives to p_2, or some consequences (that p_1 doesn't own anymore what he gave and that now p_2 owns it). This reminds the so-called "frame problem", well-known in Artificial Intelligence (McCarthy & Hayes, 1969). This information is automatically included in iconic schemata.

The model can be extended also to different problems, where other arithmetic operations are used. Beyond problems of "splitting and joining parts", also problems requiring fractions may be considered (where sets of different nature, e.g. marbles and children are numbered and put in relation).

Depending on expertise or individual strategy, iconic frames may be developed in one sense (abstract) or in the other (concrete): in one case, the set of Mjs may become a more abstract symbol, something like **marbles (Joe).** In the other case, counting or other concrete strategies can be called to solve questions about quantity, being more or less, etc.

The model can account for specific difficulties and errors. For example, it has been repeatedly found that compare problems are more difficult than other arithmetic problems (cfr. Stern, 1993): the iconic representation simulation suggests that the reason could be because a set must be broken with the only purpose of comparing parts, and this is more arbitrary (and difficult to represent) than situations where parts are really taken away.

Conclusion

In the present paper the analogical-digital distinction has been examined from the psychological standpoint in the context of mathematical problem solving. It has been argued that, even if this distinction applies both to concrete-graphic presentations and mental-iconic *re*presentations, the differences should be made clear. In the first place, concrete presentations cannot be considered direct copies of corresponding mental representations, which seem best considered only views on the former, resulting by a particular process of "reading". More importantly, in the case of concrete presentations, the difference between analogical and digital depends on how well such presentations can reveal analogies with real states of affairs, whereas digital presentations are arbitrary. In the case of iconic representations, the analogical-digital contrast depends on the level of processing (the analogical being closer to a sensorial side and the digital closer to a higher-level processing). A depicted dog's image is an analog presentation, since it is more similar to a real dog than the word "dog", but a mental dog's image is an analog representation since it is more based on perceptual elements (called "icons") than a symbolic (digital) token for "dog" for example used in reasoning about dogs. Iconic representation is opposed to propositional representation because digital tokens are usually connected in propositions, in natural or artificial languages, to express relationships between concepts.

Mental representations relevant in mathematical knowledge (mathematical situations and problems) have been then focused, and some knowledge representation systems have been described in order to discuss how well they can capture the analogic (iconic) and digital (propositional) aspects of natural representation. Frames, in particular, seem best suited to modeling mathematical representation. A simulation model that uses frames for representing simple arithmetical situations has been described, and some shortcomings of a pure propositional representation system have been shown. Then, how a frame representation system can be adapted to express iconic representations has been discussed.

In the proposed model, comprehension comes through the construction and processing in workmemory of representations at different levels. The main feature of the model is a representation formalism in which propositional and iconic elements are integrated.

Iconic frames are used like traditional frames, but slots are filled with tokens not for sets but for set elements. These representations are not concrete manipulations and the number of representation elements is not relevant. They are similar to mental models, being half-way between only propositional (or abstract) and only concrete representations of sets; however, important differences with mental models have been shown.

A main theoretical assumption of the model is that basic representations are neither fully concrete nor fully abstract. Time, number, ownership and similar, are propositional elements; the ideas of comparing, transferring, etc. are representable in a more concrete form.

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